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Diffraction at skew incidence by an anisotropic impedance wedge in electromagnetism theory: a new class of canonical cases

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Abstract. An original mathematical approach for the diffraction of an electromagnetic skew-incident wave by a wedge allows us to reduce such a problem to new elementary functional equations. Using this method, we present an exact analytical solution with closed form expressions for a class of wedge of any angle with a certain type of anisotropic boundary condition which does not require any field component to vanish on the faces. We thereby consider the important case where the principal axes of anisotropy are along and normal to the edge, the relative impedance matrix attached to each face being diagonal with its determinant equal to unity.

1. Introduction

The research of a complete explicit solution for the diffraction of an electromagnetic skew-incident plane wave by a wedge with anisotropic impedance boundary conditions is a very delicate mathematical problem: instead of being uncoupled as in the case of isotropic boundary conditions and normal incidence on the edge (Bernard 1987) the components of the electric and magnetic fields parallel to the edge can now be coupled by two equations by face as in Vaccaro (1980), Bernard (1990a) and Lyalinov (1994). In that case, the expression of the field with Sommerfeld–Maliuzhinets integrals amounts to dealing with vector functional equations for two unknown analytical spectral functions. As already noted for the Wiener–Hopf equations (see Jones (1991)), the difficulty then comes from the fact that the method developed for the scalar case (see Maliuzhinets (1958a)) cannot be used for the vector case which involves matrices and thus no commutative algebras. Specific approaches are then necessary for this case. The method of Vaccaro (1980) is suitable when the field components along the normal to *one* of the wedge faces can be used independently to express boundary conditions on *both* faces, which implies some particular choices of geometry as the half-plane, the plane discontinuity, or the right-angled wedge. In other respects, we have also the method of Lyalinov (1994) who approaches the solution of coupled functional equations by an original iterative method, which is all the more convergent as the coupling becomes weaker. Here, we choose to use the method developed in Bernard (1990a) for wedge problems at skew incidence, in order to analyse the case of a wedge with any angle and anisotropic impedance boundary conditions requiring no field component to vanish on the faces. As indicated in Bernard (1995), we consider the important category of wedges whose principal axes of anisotropy are along the edge and normal to it; an exact explicit expression of the solution is then obtained when the determinant of the diagonal impedance matrix attached to each face is equal to unity. Let us note that this case of anisotropy corresponds to reflection coefficients in geometrical optics, which are independent of the polarization of the incident field when the plane of incidence is perpendicular to the edge (normal incidence).

After we express the general problem in terms of coupled functional equations on Maliuzhinets spectral functions in section 2, and reduce it by our method in section 3, we exhibit a new class of canonical cases in section 4 with closed-form expressions of spectral functions. We give some details concerning the decomposition of the resulting field. Certain demonstrations are given in the appendices to maintain the clarity of the paper.

Before proceeding, let us mention some reasons why the determination of the analytical solution, for such a simply shaped object with an edge, is important. First physically, these solutions can help to interpret more easily the diffraction mechanisms induced by the edge. Secondly numerically, they can be used not only when a radiation is defined as the summation of elementary object contributions or when a perturbation method uses them as a base, but also to test or to hybridize pure numerical schemes of calculus. Thirdly, from a mathematical viewpoint, this investigation may open a new way for the analysis of a class of complex well-posed mathematical problems.

2. Position of the problem

2.1. Representation of the field and general properties

The geometry of the wedge is here defined in cylindrical coordinates (ρ, φ, z) by $|\varphi| \geqslant \Phi$, with the edge parallel to the z-axis (figure 1). An incoming plane wave, with a harmonic time dependence $e^{i\omega t}$ from now on assumed and suppressed throughout, illuminates the wedge. This incident field is characterized by the z-components of the electric and magnetic fields (see appendix 1), respectively E_z^i and H_z^i, given by

$$\left| \begin{matrix} E_z^i(\rho, \varphi, z) \\ H_z^i(\rho, \varphi, z) \end{matrix} \right. = \left| \begin{matrix} D_1 \\ D_2/Z_o \end{matrix} \right. e^{ik[\rho \sin\beta \cos(\varphi - \varphi') - z \cos\beta]}. \tag{1}$$

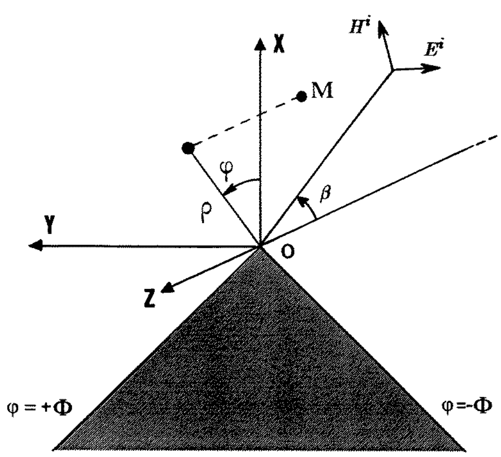

Figure 1. Geometry of the wedge.

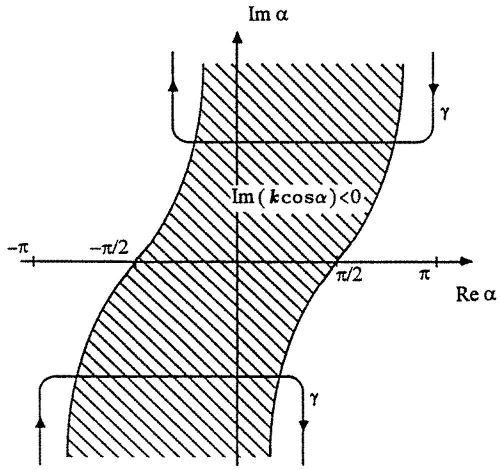

Figure 2. Contour of integration γ for complex k.

In expression (1), $k(\operatorname{Im} k \leqslant 0)$ and Z_o denote respectively the exterior medium wavenumber and impedance, β is the angle of the incident direction with the edge of the wedge.

All electrical characteristics are assumed to be independent of z. The expressions of the z-components of the electric and magnetic fields, respectively E_z and H_z, from which derives all the field satisfying Maxwell equations (see appendix 1), can be sought, for $|\varphi| \leqslant \Phi$, in the form of Sommerfeld–Maliuzhinets integrals (Maliuzhinets 1958b), as follows:

$$\left| \begin{matrix} E_z(\rho,\varphi,z) \\ H_z(\rho,\varphi,z) \end{matrix} \right. = \frac{\mathrm{e}^{-\mathrm{i}kz\cos\beta}}{2\pi\mathrm{i}} \int_\gamma \left| \begin{matrix} f_1(\alpha+\varphi) \\ f_2(\alpha+\varphi)/Z_o \end{matrix} \right. \mathrm{e}^{\mathrm{i}k\rho\sin\beta\cos\alpha}\,\mathrm{d}\alpha. \tag{2}$$

Here, $f_{1,2}$ are analytic functions and the odd path γ (see figure 2) consists of two branches: one, named γ_+, going from $(\mathrm{i}\infty + \arg(\mathrm{i}k) + (a_1 + \pi/2))$ to $(\mathrm{i}\infty + \arg(\mathrm{i}k) - (a_2 + \pi/2))$ with $0 < a_{1,2} < \pi$, as $\operatorname{Im}\alpha \geqslant d$, above all the singularities of the integrand (possible from subsequent condition (c)), and the other, named γ_-, obtained by the inversion of γ_+ with respect to $\alpha = 0$. From its form, the expression given in (2) verifies the wave equation $(\Delta + k^2)u = 0$, with Δ the Laplacian operator.

From Maliuzhinets (1958), Tuzhilin (1973a) and Bernard (1994, 1995), we can assume some general properties of the field:

(a′) the only incoming plane wave is the incident field;

(b′) every field component is locally summable with respect to ρ and each z-component tends to a finite value independent of φ in the vicinity of the edge;

(c′) the field, except possibly its geometrical optics part when $\operatorname{Im} k \neq 0$, does not grow at infinity.

The first condition (a′) is satisfied when:

(a) $[f_j(\alpha) - D_j/(\alpha - \varphi')]$ is regular for $|\operatorname{Re}\alpha| \leqslant \Phi$ $(j = 1, 2)$.

The second one (b′) is verified when (see appendix 2):

(b) some constants $g_j^{\pm}$ and some analytic function h_j exist, such as $|f_j(\alpha+\varphi) \mp f_j(-\alpha+\varphi) - g_j^{\pm}| < |h_j(\alpha)|$ on and within some γ_+, when $|\operatorname{Re}\varphi| \leqslant \Phi$, the function h_j being summable on the loop γ_+ and regular on and within it $(j = 1, 2)$.

The third condition (c′) means:

(c) $f_j(\alpha + \varphi)$ has no singularity (except possibly those corresponding to plane waves coming from infinity and attached to the incident and reflected fields) in the zone defined by $\mathrm{Re}\,(\mathrm{i}k\cos\alpha) > 0$ as $|\mathrm{Re}\,\alpha| < \pi$, $|\mathrm{Re}\,\varphi| \leqslant \Phi (j = 1, 2)$.

These conditions will be particularly useful to obtain the uniqueness of the solution in our case.

2.2. Boundary conditions and vectorial functional equations on the f_j

We consider an anisotropic impedance boundary condition on each face of the wedge $\varphi = \pm\Phi$. The electric field E and the magnetic field H are then related by $(E - n^{\pm}(n^{\pm}E)) = (Z^{\pm})(n^{\pm} \wedge H)$, where $n^{\pm}$ is the unit vector along the outward-pointing normal to the face, and $(Z^{\pm})/Z_{\mathrm{o}}$ the relative impedance operator. This type of boundary condition commonly used in scattering theory can be constructed to recover closely the geometrical reflection coefficients at any incidence. Assuming high permittivity and permeability characteristics of each face, the operator $(Z^{\pm})/Z_{\mathrm{o}}$ can be approximated by a constant tensor as in Lyalinov (1994). Letting the tangential vectors in ρ and z components, we then consider the case where $(Z^{\pm})$ is given by a diagonal constant matrix. Expressing all components of the fields as functions of E_z and H_z (see appendix 1), we can write

$$\begin{aligned} &\left[\mp\cos\beta\frac{\partial E_z}{\partial\rho} \mp \frac{\partial(Z_{\mathrm{o}}H_z)}{\rho\partial\varphi} - \mathrm{i}k\eta_h^{\pm}(\sin^2\beta)Z_{\mathrm{o}}H_z\right]\Bigg|_{\varphi=\pm\Phi} = 0 \\ &\left[\mp\frac{\partial E_z}{\rho\partial\varphi} - \mathrm{i}k(\sin^2\beta)E_z/\eta_e^{\pm} \pm \cos\beta\frac{\partial(Z_{\mathrm{o}}H_z)}{\partial\rho}\right]\Bigg|_{\varphi=\pm\Phi} = 0 \end{aligned} \tag{3}$$

where the $\eta_{e,h}^{\pm}$ are the diagonal elements of the relative impedance matrix following $(Z^{\pm})_{11}/Z_{\mathrm{o}} = \eta_h^{\pm}$, $(Z^{\pm})_{22}/Z_{\mathrm{o}} = \eta_e^{\pm}$. A condition of strict passivity is assumed, implying $\mathrm{Re}\,(\eta_{e,h}^{\pm}) > 0$ so that the geometrical optics reflection coefficients are of modulus inferior to unity. Besides, we note that if $\eta_e^{\pm} = \eta_h^{\pm}$, we recover the usual condition for isotropic constant impedance (also called the Leontovich boundary condition).

We can use the integral expressions of E_z and H_z given in (2) to write the impedance boundary conditions on each face $\varphi = \pm\Phi$. By differentiation of (2) and integration by parts, we then obtain

$$\int_{\gamma} \mathrm{d}\alpha \left[A_{\alpha}^{\pm}\left|\begin{matrix} f_1(\alpha\pm\Phi) \\ f_2(\alpha\pm\Phi)\end{matrix}\right. - A_{-\alpha}^{\pm}\left|\begin{matrix} f_1(-\alpha\pm\Phi) \\ f_2(-\alpha\pm\Phi)\end{matrix}\right.\right]\mathrm{e}^{\mathrm{i}k\rho\sin\beta\cos\alpha} = 0 \tag{4}$$

where $A_{\alpha}^{\pm}$ is a matrix function of α, given by

$$A_{\alpha}^{\pm} = \begin{bmatrix} \cos\beta\cos\alpha & \sin\alpha \pm \eta_h^{\pm}\sin\beta \\ \sin\alpha \pm \sin\beta/\eta_e^{\pm} & -\cos\beta\cos\alpha \end{bmatrix}. \tag{5}$$

As in Bernard (1990a), we use an inversion theorem (see appendix 3) due to Maliuzhinets (1958b), which we can apply when the term within brackets in (4) is $\mathrm{O}(\mathrm{e}^{\tau|\mathrm{Im}\,\alpha|})$ as $|\mathrm{Im}\,\alpha| \to \infty$, τ being a constant. In our case, the condition (b) is satisfied, which implies $f(\alpha) = \mathrm{O}(1)$ and $|\mathrm{Im}\,\alpha| \to \infty$, and so $\tau = 1$. This gives us

$$A_{\alpha}^{\pm}\left|\begin{matrix} f_1(\alpha\pm\Phi) \\ f_2(\alpha\pm\Phi)\end{matrix}\right. - A_{-\alpha}^{\pm}\left|\begin{matrix} f_1(-\alpha\pm\Phi) \\ f_2(-\alpha\pm\Phi)\end{matrix}\right. = \sin\alpha\sum_{1\leqslant n\leqslant\tau}(\cos\alpha)^{n-1}\left|\begin{matrix} c_{1n} \\ c_{2n}\end{matrix}\right. = \sin\alpha\left|\begin{matrix} c_1 \\ c_2\end{matrix}\right. \tag{6}$$

with constant numbers c_1, c_2. Let us note that the system of vectorial functional equations (6) extends the meromorphy of the functions f_j $(j = 1, 2)$, that was initially assumed in the band $|\mathrm{Re}\,\alpha| \leqslant \Phi$ from (a), to the whole complex plane.

Note. In the case of some more general boundary conditions, where the term within brackets in (4) would be $\mathrm{O}(\mathrm{e}^{\tau|\mathrm{Im}\,\alpha|})$ as $|\mathrm{Im}\,\alpha| \to \infty$ with τ *arbitrary*, the constants c_{1n}, c_{2n} could be necessary in order to satisfy supplementary conditions at the edge, or to satisfy the condition (c). However, in our case, with $A_\alpha^\pm$ given by (5), it is interesting to note that we could add to the f_j some constants (permitted from the oddness of γ) in order to make the $c_{1,2}$ vanish.

3. Reduction of the problem

We will now adapt the general procedure defined in Bernard (1990a) to reduce the coupled functional equations (see also Bernard (1995)). The heart of the matter is to modify (6), in a way which preserves the equivalence with the initial system of equations.

In our method, we search for an even or odd linear operator $B_\alpha^\pm$ so that we can write $B_\alpha^\pm A_\alpha^\pm = C_{\alpha\pm\Phi}$ with $B_{-\alpha}^\pm = \varepsilon^\pm B_\alpha^\pm$, $(\varepsilon^\pm)^2 = 1$. This gives us, when $B_\alpha^\pm$ is applied to (6),

$$\begin{vmatrix} t_1(\alpha\pm\Phi) \\ t_2(\alpha\pm\Phi) \end{vmatrix} - \varepsilon^\pm \begin{vmatrix} t_1(-\alpha\pm\Phi) \\ t_2(-\alpha\pm\Phi) \end{vmatrix} = B_\alpha^\pm \sin\alpha \begin{vmatrix} c_1 \\ c_2 \end{vmatrix} \tag{7}$$

with $\begin{vmatrix} t_1(\alpha) \\ t_2(\alpha) \end{vmatrix} = C_\alpha \begin{vmatrix} f_1(\alpha) \\ f_2(\alpha) \end{vmatrix}$. We then have a system of two *independent* sets of Maliuzhinets-type equations on the t_j $(j = 1, 2)$. The general solution of this type of equations on each function t_j is known (Maliuzhinets, Tuzhilin, Bernard) when the considered function is *meromorphic* in the strip $|\mathrm{Re}\,\alpha| \leqslant \Phi$, with a finite set of poles in this band. Now, since the functions f_j have also to satisfy, from (a), the property of *meromorphy* in the band $|\mathrm{Re}\,\alpha| \leqslant \Phi$, it is reasonable to search for a matrix C_α satisfying it too. The matrix C_α has, besides, to be inversible in order to recover in a unique manner the functions f_j from the t_j. Note that we do not need to specify in advance the regularity and behaviour at infinity of the matrix C_α. Once C_α is chosen, the properties (a)–(c) of the functions f_j will give us those satisfied by the t_j. This will imply (as we will see later for the new canonical case exhibited) the uniqueness of choice for the t_j and hence for the complete solution.

Thus, the main problem is now to find C_α as previously defined, which is equivalent to searching for $(C_\alpha)^{-1}$ such as

$$(A_\alpha^\pm)(C_{\alpha\pm\Phi})^{-1} - \varepsilon^\pm(A_{-\alpha}^\pm)(C_{-\alpha\pm\Phi})^{-1} = 0 \tag{8a}$$

or, by multiplication with $\varepsilon^\pm C_{\alpha\pm\Phi}(A_\alpha^\pm)^{-1}$:

$$(C_{\alpha\pm\Phi})[(A_\alpha^\pm)^{-1}(A_{-\alpha}^\pm)](C_{-\alpha\pm\Phi})^{-1} = \varepsilon^\pm \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{8b}$$

In (8), $(C_\alpha)^{-1}$ has to be meromorphic as $|\mathrm{Re}\,\alpha| \leqslant \Phi$, with $\det((C_\alpha)^{-1}) \neq 0$ (the zero function) so that C_α remains defined, and $(\varepsilon^\pm)^2 = 1$. We note, that as $A_\alpha^\pm$ is meromorphic, the previous equation (8*a*) extends the meromorphy of C_α to the whole complex plane (the set of poles of C_α is completely defined from (8*a*) and the condition (a).

Then, we let

$$(C_\alpha)^{-1} = \begin{bmatrix} b(\alpha) & -c(\alpha) \\ -d(\alpha) & a(\alpha) \end{bmatrix}. \tag{9}$$

From now on, for the sake of simplicity,

$$a_+ = a(\alpha\pm\Phi) \qquad a_- = a(-\alpha\pm\Phi)$$

and so on for b, c, d; the superscript index of face $\varphi = \pm\Phi$ can be omitted for all quantities.

We can write the equation (8*b*) in the following form:

$$\frac{1}{\det(A_\alpha)}\begin{bmatrix} -c_+b_-g\left(\dfrac{a_+}{c_+}, \dfrac{d_-}{b_-}\right) & c_+c_-g\left(\dfrac{a_+}{c_+}, \dfrac{a_-}{c_-}\right) \\ b_+b_-g\left(\dfrac{d_+}{b_+}, \dfrac{d_-}{b_-}\right) & c_-b_+g\left(\dfrac{d_+}{b_+}, \dfrac{a_-}{c_-}\right) \end{bmatrix} = \begin{bmatrix} \varepsilon & 0 \\ 0 & \varepsilon \end{bmatrix}(a_+b_+ - c_+d_+) \tag{10a}$$

with

$$g(r,s) = n(\alpha).r.s - (l(\alpha).r - l(-\alpha).s) - p(\alpha) \tag{10b}$$

$$(A_\alpha)^{-1}(A_{-\alpha}) = \begin{bmatrix} l(\alpha) & n(\alpha) \\ p(\alpha) & l(-\alpha) \end{bmatrix} \frac{1}{\det A_\alpha} \tag{10c}$$

where we remark that n and p are odd functions.

When n is not zero (i.e. the coupled case), c and b cannot be equal to the function zero (see appendix 5). Thus, we can rewrite the nondiagonal terms of (10a) as

$$g\left(\frac{a_+}{c_+}, \frac{a_-}{c_-}\right) = 0 \tag{11a}$$

$$g\left(\frac{d_+}{b_+}, \frac{d_-}{b_-}\right) = 0 \tag{11b}$$

with $a/c \neq d/b$ (since $\det(C_\alpha)^{-1} \neq 0$).

Then, we can use the two remaining equations of (10a). After some manipulations detailed in appendix 4, it is found that the conditions on the diagonal terms in (10a) are strictly equivalent to the following equalities

$$\left(\frac{b_+}{c_+}\right)\left(\frac{b_-}{c_-}\right)^{-1} = \frac{-g(a_+/c_+, d_-/b_-)}{g(d_+/b_+, a_-/c_-)} \tag{11c}$$

$$\frac{(b_+a_+)}{(b_-a_-)} = \frac{\det(A_{-\alpha})}{\det(A_\alpha)} \frac{1-(c_-/a_-)(d_-/b_-)}{1-(c_+/a_+)(d_+/b_+)} \tag{11d}$$

when (11a) and (11b) are satisfied. Equations (11c) and (11d) can be solved after their second member is defined, i.e. once a/c, d/b solutions of (11a) and (11b) are determined. So defined, the set of equations (11a–d) can be particularly efficient.

4. The closed-form solution for the case $\eta_h^\pm \eta_e^\pm = 1$

4.1. The exact analytical expression of the f_j ($j = 1, 2$)

We now choose to solve in closed form the important case where the relative impedance coefficients satisfy $\eta_h^\pm \eta_e^\pm = 1$ with $\mathrm{Re}\,(\eta_{e,h}^\pm) > 0$ (see Bernard (1995)), which is often named the balanced hybrid condition. This condition is well known in the theory of horns (Lier *et al* 1987) and absorbers (Yee and Chang 1991). In this case, the geometrical optics reflection coefficients are independent of the polarization at normal incidence to the edge.

As indicated previously, we will seek the elements of $(C_\alpha)^{-1}$ and then determine the expressions of the f_j ($j = 1, 2$).

4.1.1. Determination of (C_α^{-1}). We now solve the equations (11a–d) which concern the elements of (C_α^{-1}). We need at first to define the elements of (10c). By elementary calculus on $A_\alpha^\pm$, we have

$$l^\pm(\alpha) = \cos^2\beta \sin^2\alpha - \cos^2\alpha - (\eta_h^\pm/\eta_e^\pm - 1)\sin^2\beta \tag{12a}$$

$$n(\alpha) = -p(\alpha) = \cos\beta \sin(2\alpha) \tag{12b}$$

$$\det(A_\alpha^\pm) = -\left(\sin\alpha\sin\beta \pm \left(\eta_h^\pm + \cos\beta\sqrt{(\eta_h^\pm)^2 - 1}\right)\right) \times \left(\sin\alpha\sin\beta \pm \left(\eta_h^\pm - \cos\beta\sqrt{(\eta_h^\pm)^2 - 1}\right)\right) \tag{12c}$$

where $(\eta_h^{\pm} + \epsilon \cos\beta\sqrt{(\eta_h^{\pm})^2 - 1}) = \sin(\beta) \times \sin(\theta_1^{\pm} - \epsilon \mathrm{i}\delta)$ with $\eta_h^{\pm} = \sin\theta_1^{\pm}$ $(0 < \mathrm{Re}\,\theta_1^{\pm} \leqslant \pi/2)$, $\delta = \ln(\tan(\beta/2))$, $\epsilon = +$ or -1.

As the function $l(\alpha)$ is an even function, it is possible, for the treatment of (11a) and (11b), to put

$$\frac{a(\alpha)}{c(\alpha)} = \tan(v_{an}(\alpha)) \qquad \frac{d(\alpha)}{b(\alpha)} = \tan(v_{an}(\alpha) + \pi/2) \tag{13}$$

which gives us a (necessary and sufficient) condition on v_{an},

$$v'_{an}(\alpha \pm \Phi) + v'_{an}(-\alpha \pm \Phi) = \left(\arctan\left(\frac{n(\alpha)}{l^{\pm}(\alpha)}\right)\right)' \tag{14}$$

where $(.)'$ denotes the derivative of the function (.). From the use of the properties of Fourier transform as in Maliuzhinets (1958b), Vaccaro (1980) and Bernard (see appendix 6), a regular solution of (14) for $|\mathrm{Re}\,\alpha| < \Phi$, with $v'_{an}(\alpha) = \mathrm{O}(1)$ as $|\mathrm{Im}\,\alpha| \to \infty$, is possible in a double integral form:

$$v'_{an}(\alpha) = \frac{\mathrm{i}}{\sqrt{2\pi}}\,\mathrm{v.p.}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\left(\frac{R_+(\omega)\mathrm{e}^{-\mathrm{i}\omega\Phi}}{\mathrm{i}\sin(2\omega\Phi)} - \frac{R_-(\omega)\mathrm{e}^{\mathrm{i}\omega\Phi}}{\mathrm{i}\sin(2\omega\Phi)}\right)\mathrm{e}^{-\mathrm{i}\omega\alpha}\,\mathrm{d}\omega \tag{15a}$$

with

$$R_{\pm}(\omega) = \frac{-\mathrm{i}}{2\sqrt{2\pi}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} S^{\pm}(\alpha')\mathrm{e}^{\mathrm{i}\omega\alpha'}\,\mathrm{d}\alpha' \tag{15b}$$

and $S^{\pm}(\alpha) = (\arctan(n(\alpha)/l(\alpha)))'$. In (15$a$) the term v.p. signifies that we take the principal value of the integral. We obtain a closed-form expression of $R_{\pm}$ by the method of residues, which yields, for $|\mathrm{Re}\,\alpha| < \Phi$,

$$\begin{aligned} v_{an}(\alpha) = \mathrm{i}\sum_{\pm}\sum_{\epsilon=+1,-1}&\left(\epsilon\int_0^{\infty}\frac{\mathrm{e}^{-\nu(\theta_1^{\pm}-\epsilon\mathrm{i}\delta)} - \mathrm{e}^{-\nu(\pi-\theta_1^{\pm}+\epsilon\mathrm{i}\delta)}}{2\times(1-\mathrm{e}^{-\nu\pi})}\right.\\ &\left.\times\frac{\mp(\cosh(\nu(\alpha\pm\Phi))-1)}{\nu\sinh(2\nu\Phi)}\,\mathrm{d}\nu\right) + v_{\mathrm{o}} \end{aligned} \tag{16a}$$

v_{o} being an arbitrary constant. Now, we can use the expansion

$$(1-x)^{-1} = \sum_{l=0}^{N} x^l + x^{(N+1)}/(1-x)$$

with the expression, obtained from Gradshteyn (form. 3.541–2),

$$\int_0^{\infty}\mathrm{e}^{-\mu x}\frac{\cosh(\beta x)-1}{x\sinh(bx)}\,\mathrm{d}x = \ln\left(\frac{\Gamma\left(\frac{1}{2}+\frac{1}{2b}(\mu+\beta)\right)}{\Gamma\left(\frac{1}{2}+\frac{\mu}{2b}\right)} \times \frac{\Gamma\left(\frac{1}{2}+\frac{1}{2b}(\mu-\beta)\right)}{\Gamma\left(\frac{1}{2}+\frac{\mu}{2b}\right)}\right)$$

to expand the previous integral expression of v_{an} with Γ being the gamma function. By doing so, we obtain the following efficient expression for $|\mathrm{Re}\,\alpha| < \Phi + (N+1)\pi$,

$$\begin{aligned} v_{an}(\alpha) = \mathrm{i}\sum_{\pm}\sum_{\epsilon=+1,-1}&\left(\sum_{l=0}^{N}\ln\left(\left[a_l \times \Gamma\left(\frac{1}{2}+\frac{1}{4\Phi}(-(\alpha\pm\Phi)+\theta_1^{\pm}-\epsilon\mathrm{i}\delta+l\pi)\right)^{\mp\frac{1}{2}}\right.\right.\right.\\ &\times\Gamma\left(\frac{1}{2}+\frac{1}{4\Phi}((\alpha\pm\Phi)+\theta_1^{\pm}-\epsilon\mathrm{i}\delta+l\pi)\right)^{\mp\frac{1}{2}}\\ &\times\Gamma\left(\frac{1}{2}+\frac{1}{4\Phi}((\alpha\pm\Phi)+\pi-\theta_1^{\pm}+\epsilon\mathrm{i}\delta+l\pi)\right)^{\pm\frac{1}{2}} \end{aligned}$$

$$\times\Gamma\left(\frac{1}{2}+\frac{1}{4\Phi}(-(\alpha\pm\Phi)+\pi-\theta_1^\pm+\epsilon\mathrm{i}\delta+l\pi)\right)^{\pm\frac{1}{2}}\Bigg]^\epsilon\Bigg)$$
$$+\epsilon\int_0^\infty\frac{\mathrm{e}^{-\nu(\theta_1^\pm-\epsilon\mathrm{i}\delta)}-\mathrm{e}^{-\nu(\pi-\theta_1^\pm+\epsilon\mathrm{i}\delta)}}{2\times(1-\mathrm{e}^{-\nu\pi})}$$
$$\times\mathrm{e}^{-\nu(N+1)\pi}\frac{\mp(\cosh(\nu(\alpha\pm\Phi))-1)}{\nu\sinh(2\nu\Phi)}\,\mathrm{d}\nu\Bigg)+v_\mathrm{o} \tag{16b}$$

with

$$a_l=\Gamma\left(\frac{1}{2}+\frac{1}{4\Phi}(\theta_1^\pm-\epsilon\mathrm{i}\delta+l\pi)\right)\Bigg/\Gamma\left(\frac{1}{2}+\frac{1}{4\Phi}(\pi-\theta_1^\pm+\epsilon\mathrm{i}\delta+l\pi)\right).$$

The constant v_o can be chosen arbitrarily. So we can let $a_l=1$ and $v_\mathrm{o}=0$ in the previous expression (16*b*) (as N is finite) and then define a new solution of (14) $v_{an(,N)}$, which depends on N (the term $v_{an(,N)}(\alpha)-v_{an(,N)}(\alpha')$ does not depend on N).

Some properties of the expression (16*b*) for v_{an} can now be detailed. We note for (13) that $\exp(2\mathrm{i}v_{an})$, and then $\tan(v_{an})$ and $\mathrm{cotan}(v_{an})$, are meromorphic, even if v_{an}, given by (16*b*), is multiform. Furthermore, we can also consider the behaviour of v_{an} when $|\mathrm{Im}\,\alpha|\to\infty$. At infinity, the principal contribution for the evaluation of the above integral comes from the vicinity of $\nu=0$. Then, since we have, from Gradshtein (1980),

$$\int_0^\infty\frac{1-\cosh(\nu x)}{\nu\sinh(2\nu\Phi)}\,\mathrm{d}\nu=-\int_0^x\frac{\pi}{4\Phi}\tan\left(\frac{\pi x'}{4\Phi}\right)\mathrm{d}x'=\ln\left(\cos\left(\frac{\pi x}{4\Phi}\right)\right)\qquad\text{for }|\mathrm{Re}\,x|<2\Phi$$

we easily obtain $v_{an}(\alpha)=\mathrm{O}(1)$ as $|\mathrm{Im}\,\alpha|\to\infty$. By another way indicated in appendix 6, we obtain more precisely $v_{an}(\alpha)=A_1(1+\mathrm{O}(\mathrm{e}^{-c|\mathrm{Im}\,\alpha|}))$ as $|\mathrm{Im}\,\alpha|\to\infty$, A_1 and $c>0$ being some constants. In other respects we remark that (14) can be used with (16*b*) to easily extend the calculus of v_{an} to the whole complex plane.

The expression of v_{an} being found, we can use the fact that, from (13), $a/c=-b/d=\tan(v_{an})$ to express directly (11*c*–*d*) in a new form. We can then write

$$\left(\frac{b_+}{c_+}\right)\left(\frac{b_-}{c_-}\right)^{-1}=\left(\frac{a_+}{c_+}\right)\left(\frac{a_-}{c_-}\right)^{-1} \tag{17a}$$
$$\left(\frac{a_+b_+}{a_-b_-}\right)=\frac{\det A^\pm_{-\alpha}}{\det A^\pm_\alpha}\frac{1+(\cot(v(-\alpha\pm\Phi)))^2}{1+(\cot(v(\alpha\pm\Phi)))^2}. \tag{17b}$$

This leads us to choose $a=b$. From (13), $a/c=-b/d$, thus we have $d=-c$. We then obtain, from (17*b*),

$$a(\alpha)=\Psi_{an}(\alpha)\sin(v_{an}(\alpha))\qquad c(\alpha)=\Psi_{an}(\alpha)\cos(v_{an}(\alpha)) \tag{18}$$

where the (necessary and sufficient) condition on $\Psi_{an}(\alpha)$ is

$$\left(\frac{\Psi_{an}(\alpha\pm\Phi)}{\Psi_{an}(-\alpha\pm\Phi)}\right)^2=\frac{\det A^\pm_-\alpha}{\det A^\pm_\alpha}. \tag{19}$$

We transform (19) in a standard manner by taking the logarithmic derivative of each side to obtain a functional equation of the same type as (14),

$$\frac{\Psi'_{an}}{\Psi_{an}}(\alpha\pm\Phi)+\frac{\Psi'_{an}}{\Psi_{an}}(-\alpha\pm\Phi)=S^\pm(\alpha) \tag{20}$$

with $S^\pm(\alpha)=(\ln(\det A^\pm_{-\alpha}/\det A^\pm_\alpha))'/2$. We can solve (20) as previously done for (14). We begin by writing a double-integral expression of a solution (see appendix 6), regular for $|\mathrm{Re}\,\alpha|<\Phi$, and $\mathrm{O}(1)$ as $|\mathrm{Im}\,\alpha|\to\infty$. This expression is then reduced to a simple integral form by the method of residues. By means of Ψ'_{an}/Ψ_{an}, we then develop Ψ_{an}, for $|\mathrm{Re}\,\alpha| < \Phi$,

$$\Psi_{an}(\alpha) = \Psi_o \times \prod_{\epsilon=-1 et +1} \prod_{\pm} \left(\exp\left(\int_0^\infty \frac{\mathrm{e}^{-\nu(\theta_1^\pm - \epsilon \mathrm{i}\delta)} + \mathrm{e}^{-\nu(\pi-\theta_1^\pm+\epsilon\mathrm{i}\delta)}}{2\times(1+\mathrm{e}^{-\nu\pi})} \times \frac{1-\cosh(\nu(\alpha\pm\Phi))}{\nu\sinh(2\nu\Phi)}\,\mathrm{d}\nu \right)\right) \tag{21a}$$

where Ψ_o denotes an arbitrary constant. We now use the expansion

$$(1+x)^{-1} = \sum_{l=0}^{N} (-x)^l + (-x)^{(N+1)}/(1+x)$$

with the expression

$$\int_0^\infty \mathrm{e}^{-\mu x} \frac{\cosh(\beta x)-1}{x\sinh(bx)}\,\mathrm{d}x = \ln\left(\frac{\Gamma\left(\frac12 + \frac{1}{2b}(\mu+\beta)\right)}{\Gamma\left(\frac12+\frac{\mu}{2b}\right)} \times \frac{\Gamma\left(\frac12 + \frac{1}{2b}(\mu-\beta)\right)}{\Gamma\left(\frac12+\frac{\mu}{2b}\right)} \right)$$

to expand the integral expression with the gamma function Γ, so that we obtain an efficient expression for $|\mathrm{Re}\,\alpha| < \Phi + (N+1)\pi$ which reads as

$$\begin{aligned}\Psi_{an}(\alpha) = \Psi_o \times \prod_{\epsilon=-1 et +1} \prod_{\pm} \Bigg(\prod_{l=0}^{N} \Bigg[& b_l \times \Gamma\left(\frac12 + \frac{1}{4\Phi}((\alpha\pm\Phi)+\theta_1^\pm - \epsilon\mathrm{i}\delta + l\pi)\right)^{-\frac12} \\ & \times \Gamma\left(\frac12 + \frac{1}{4\Phi}(-(\alpha\pm\Phi)+\theta_1^\pm - \epsilon\mathrm{i}\delta + l\pi)\right)^{-\frac12} \\ & \times \Gamma\left(\frac12 + \frac{1}{4\Phi}((\alpha\pm\Phi)+\pi-\theta_1^\pm + \epsilon\mathrm{i}\delta + l\pi)\right)^{-\frac12} \\ & \times \Gamma\left(\frac12 + \frac{1}{4\Phi}(-(\alpha\pm\Phi)+\pi-\theta_1^\pm + \epsilon\mathrm{i}\delta + l\pi)\right)^{-\frac12} \Bigg]^{(-1)^l} \\ & \times \exp\left(\int_0^\infty (-1)^{N+1} \times \frac{\mathrm{e}^{-\nu(\theta_1^\pm - \epsilon \mathrm{i}\delta)} + \mathrm{e}^{-\nu(\pi-\theta_1^\pm+\epsilon\mathrm{i}\delta)}}{2\times(1+\mathrm{e}^{-\nu\pi})} \times \mathrm{e}^{-\nu(N+1)\pi} \right. \\ & \left. \times \frac{1-\cosh(\nu(\alpha\pm\Phi))}{\nu\sinh(2\nu\Phi)}\,\mathrm{d}\nu \right)\Bigg)\end{aligned} \tag{21b}$$

with

$$b_l = \Gamma\left(\frac12 + \frac{1}{4\Phi}(\theta_1^\pm - \epsilon\mathrm{i}\delta + l\pi)\right) \times \Gamma\left(\frac12 + \frac{1}{4\Phi}(\pi - \theta_1^\pm + \epsilon\mathrm{i}\delta + l\pi)\right).$$

The constant Ψ_o can be chosen arbitrarily. So we can let $b_l = 1$ and $\Psi_o = 1$ in expression (21b) (as N is finite) and then define a new solution of (20) $\Psi_{an(,N)}$, which depends on N (but the term $\Psi_{an(,N)}(\alpha)/\Psi_{an(,N)}(\alpha')$ does not depend on N).

Some properties of expression (21b) of Ψ_{an} can be discussed. This function has no zero or pole in the band $|\mathrm{Re}\,\alpha| \leqslant \Phi$, and with expression (16b) of $v_{an}(\alpha)$, the functions $\Psi_{an}(\alpha)\exp(\pm \mathrm{i} v_{an}(\alpha))$, and every elements of C_α^{-1}, are meromorphic. We can also consider the behaviour of Ψ_{an} as $|\mathrm{Im}\,\alpha| \to \infty$. At infinity, the principal contribution for the evaluation of the above integral comes from the vicinity of $\nu = 0$. Then, since we have

$$\int_0^\infty \frac{1-\cosh(\nu x)}{\nu\sinh(2\nu\Phi)}\,\mathrm{d}\nu = -\int_0^x \frac{\pi}{4\Phi}\tan\left(\frac{\pi x'}{4\Phi}\right)\mathrm{d}x' = \ln\left(\cos\left(\frac{\pi x}{4\Phi}\right)\right) \qquad \text{for } |\mathrm{Re}\,x| < 2\Phi$$

we easily derive $\Psi_{an}(\alpha) = \mathrm{O}(\cos(\mu\alpha))$ with $\mu = \pi/2\Phi$, or more precisely, from appendix 6, $\Psi_{an}(\alpha) = A_2 \cos(\mu\alpha)(1 + \mathrm{O}(\mathrm{e}^{-c|\mathrm{Im}\,\alpha|}))$, A_2 and $c > 0$ being some constants.

The principal problem of the determination of a C_α satisfying (8) and being meromorphic is thus solved with our choice ($\varepsilon^\pm = 1$ in (8), since the term $(C_{\pm\Phi})(A^\pm_{\alpha=0})^{-1}$ and its inverse are definite).

Some properties of $(C_\alpha)^{-1}$ and C_α can be detailed from the ones of Ψ_{an} and v_{an}. The matrix $(C_\alpha)^{-1}$ is regular (except at infinity) for $|\mathrm{Re}\,\alpha| \leqslant \Phi$, and meromorphic in the whole complex plane. We have $\det((C_\alpha)^{-1}) = (\Psi(\alpha))^2$, and then (C_α) is regular as its inverse for $|\mathrm{Re}\,\alpha| \leqslant \Phi$. Concerning the behaviour at infinity, we find that $(C_\alpha)^{-1} = A_3 \cos(\mu\alpha)(1+\mathrm{O}(\mathrm{e}^{-c|\mathrm{Im}\,\alpha|}))$ with $\mu = \pi/2\Phi$, A_3 and $c > 0$ being some constants.

4.1.2. Determination of the f_j. The principal properties of C_α now defined, we can deduce some properties of the functions t_j of (7), using

$$\begin{vmatrix} t_1(\alpha) \\ t_2(\alpha) \end{vmatrix} = C_\alpha \begin{vmatrix} f_1(\alpha) \\ f_2(\alpha) \end{vmatrix}$$

as the meromorphic functions f_j satisfy the properties (a) and (b). The functions t_j have to be meromorphic, regular in the band $|\mathrm{Re}\,\alpha| \leqslant \Phi$ except for one simple pole at $\alpha = \varphi'$, with $t_j(\alpha) = \mathrm{O}(1/\cos(\mu\alpha))$, $\mu = \pi/2\Phi$.

We can then specify the expressions of the t_j. It is known (Tuzhilin 1970) that $\sigma(\alpha) = \mu\cos(\mu\varphi')(\sin(\mu\alpha) - \sin(\mu\varphi'))^{-1}$ is the unique function, with the properties of t_j defined just above, which satisfies the set of homogenous equations $\sigma(\alpha \pm \Phi) - \sigma(-\alpha \pm \Phi) = 0$ (i.e. (7) without second member, with $\varepsilon^\pm = 1$) and which has a residue equal to one at the pole $\alpha = \varphi'$. The solution of (7) is then necessarily equal to the sum of two terms, namely a constant vector multiplied by $\sigma(\alpha)$, and a solution of (7) regular and $\mathrm{O}(1/\cos(\mu\alpha))$ in the strip $|\mathrm{Re}\,\alpha| \leqslant \Phi$.

The constant vector being taken so that condition (a) is satisfied, we have a unique definition of the t_j. We can then write the solution of (6) satisfying (a) and (b), as follows

$$\begin{vmatrix} f_1(\alpha) \\ f_2(\alpha) \end{vmatrix} = (C_\alpha)^{-1}(C_{\varphi'}) \begin{vmatrix} D_1 \\ D_2 \end{vmatrix} \sigma(\alpha) + (C_\alpha)^{-1}(T_\alpha) \begin{vmatrix} c_1 \\ c_2 \end{vmatrix} \tag{22}$$

where the second term on the right-hand side is the particular solution of (6) regular and $\mathrm{O}(1)$ for $|\mathrm{Re}\,\alpha| \leqslant \Phi$. As can be seen by inspection of (6), this latter term is, in fact, equal to a constant vector which can be set to zero because of the oddness of γ.

Now that the expression of the f_j has been defined, it seems interesting to specify some elements concerning the conditions (b) and (c) for these functions. About (b), we observe, from the behaviours at infinity of Ψ_{an} and v_{an} previously determined, that the f_j satisfy some stronger condition than (b) consisting of: $|f_j(\alpha) - f_j(\pm\mathrm{i}\infty)| < B/\cosh(c\,\mathrm{Im}\,\alpha)$ as $\mathrm{Im}\,\alpha \to \pm\infty$, with B and c being some strictly positive constants. One can also verify with (22) that the condition (c) is satisfied. Because the matrix $(C_\alpha)^{-1}$ is analytic in the strip $|\mathrm{Re}\,\alpha| \leqslant \Phi$, we can deduce all its poles from (8) and from the knowledge of the zeros of $\det A^\pm_\alpha$ (given by (12c)). Consequently, we find that $(C_\alpha)^{-1}$ has no pole in the band $|\mathrm{Re}\,\alpha| \leqslant \pi + \Phi$ (more precisely in the strip $|\mathrm{Re}\,\alpha| < \Phi + \mathrm{Re}\,(\pi + \theta_1^\pm - \mathrm{i}\epsilon\delta)$ with $\mathrm{Re}\,\theta_1^\pm > 0$). Moreover, as $\sigma(\alpha)$ contains only poles of geometrical optics (these poles, depending only on Φ and the incident angle φ', are detailed in section 4.2), the condition (c) is verified.

Developing the expression (22), we have finally the exact closed-form expression:

$$\begin{bmatrix} f_1(\alpha) \\ f_2(\alpha) \end{bmatrix} = \frac{\Psi_{an}(\alpha)}{\Psi_{an}(\varphi')} \begin{bmatrix} \cos\Delta(\alpha) & \sin\Delta(\alpha) \\ -\sin\Delta(\alpha) & \cos\Delta(\alpha) \end{bmatrix} \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} \sigma(\alpha) \tag{23}$$

where $\Delta(\alpha) = v_{an}(\alpha) - v_{an}(\varphi')$.

It should be noted that, at normal incident $(\beta = \pi/2)$, $v_{an}(\alpha)$ is a constant and the expression given by Maliuzhinets (1958a) is recovered. Referring to Bernard (1990a, b), the reader will also easily verify the isotropic limit case with $\eta_h^\pm = \eta_e^\pm = 1$.

4.2. About the decomposition of the field

The integral expressions of the z-components of the field are now explicitly defined by (23) and (2) but it is interesting to show how to develop them, as in Maliuzhinets (1958a), Tuzhilin (1973a), Bernard (1987) and Lyalinov (1994), in order to exhibit the terms of geometrical optics fields (incident field, reflected fields) and the terms specifically excited by the edge (leaky waves, and edge diffracted wave).

We then deform the integration path γ to a steepest descent path (SDP), summing the contributions of poles encountered during this deformation. The contour SDP consists of two branches SDP $_\pm$, respectively centred on $\pm\pi$, satisfying the equation $\mathrm{Im}\,(\mathrm{i}k(\cos\alpha+1)) = 0$ with $\mathrm{i}k(\cos\alpha+1) \leqslant 0$ i.e. for k real positive, $\mathrm{Re}\,(\alpha \mp \pi)/2 = \arctan(\tanh(\mathrm{Im}\,(\alpha \mp \pi)/2))$. For the sake of simplicity, we now consider $\Phi \geqslant \pi/2$ and φ' real. So, taking E_z for example, we have

$$E_z(\rho,\varphi,z) = E_{zi} + \Sigma E_{2r}^\pm + \Sigma E_{zs,\epsilon}^\pm + \frac{\mathrm{e}^{-\mathrm{i}k(z\cos\beta+\rho\sin\beta)}}{2\pi\mathrm{i}} \int_{\mathrm{SDP}} f_1(\alpha+\varphi)\mathrm{e}^{\mathrm{i}k\rho\sin\beta(\cos\alpha+1)}\,\mathrm{d}\alpha \tag{24}$$

where:

- the term E_{zi} is related to the contribution of the pole $\alpha = \varphi'$ of $\sigma(\alpha)$:

$$E_{zi} = U(\pi - |\varphi' - \varphi|)\mathrm{e}^{-\mathrm{i}k(z\cos\beta - \rho\sin\beta\cos(\varphi'-\varphi))}$$

which is the incident field in the illuminated region, and zero in the shadow zone. U is the Heaviside step function;

- the terms $E_{zr}^\pm$ are the fields reflected by the faces $\varphi = \pm\Phi$, and related to the other poles of $\sigma(\alpha)$ with $|\mathrm{Re}\,\alpha| \leqslant \pi + \Phi$:

$$E_{zr}^\pm = U(\pi - |\pm 2\Phi - (\varphi'+\varphi)|)\mathrm{e}^{-\mathrm{i}k(z\cos\beta - \rho\sin\beta\cos(\pm 2\Phi - (\varphi'+\varphi)))} \times \mathrm{Res}(f_1(\alpha+\varphi))|_{\alpha+\varphi = \pm 2\Phi - \varphi'}$$

- $E_{zs,\epsilon}^\pm$ are the nonuniform leaky waves terms with complex phase functions, related to the poles $\alpha_{s,\epsilon}^\pm$ of $(C_\alpha)^{-1}$ in the band $|\mathrm{Re}\,\alpha| < 3\pi/2 + \Phi$. As $(C_\alpha)^{-1}$ is regular for $|\mathrm{Re}\,\alpha| \leqslant \Phi$, these poles are found easily from (8), knowing the zeros of $\det A_{\pm\alpha}^\pm$ with positive real parts given from (12*c*) as $(\pi + \theta_1^\pm - \mathrm{i}\epsilon\delta)$ for $\epsilon = +1, -1$. We have then $\alpha_{s,\epsilon}^\pm = \pm(\pi + \Phi + \theta_1^\pm - \mathrm{i}\epsilon\delta)$ with $\epsilon = +1$ and -1, and

$$E_{zs,\epsilon}^\pm = U(\pm\varphi - \varphi_s^\pm)\mathrm{e}^{-\mathrm{i}k(z\cos\beta - \rho\sin\beta\cos(\alpha_{s,\epsilon}^\pm - \varphi))}\mathrm{Res}(f_1(\alpha+\varphi))|_{\alpha+\varphi = \alpha_{s,\epsilon}^\pm}$$

where U is the Heaviside function ($U(x) = 1$ as $x > 0$, $U(x) = 0$ as $x < 0$), and $\varphi_s^\pm = \Phi + \mathrm{Re}\,\theta_1^\pm - 2\arctan(\tanh(\mathrm{Im}\,(\theta_1^\pm - \mathrm{i}\epsilon\delta)/2))$ when k is real positive. These waves are generally speaking attenuating waves passing along the faces to infinity (see Tamir and Oliner (1963)). Under the condition $\mathrm{Re}\,\theta_1^\pm = 0$, $\mathrm{Im}\,(\theta_1^\pm - \mathrm{i}\epsilon\delta) > 0$, these waves remain unattenuated. Note that $E_{zs,\epsilon}^\pm = 0$ for $\varphi_s^\pm \geqslant \Phi$, in particular for the isotropic case where $\theta_1^\pm = \pi/2$;

- the last term in (24) is principally radiated conically from the edge as $\rho \to \infty$. Approximating $f_1(\alpha+\varphi)$ on SDP$_\pm$ by its value at the saddle points $\alpha = \pm\pi$ then gives us

$$E_{ze} = \frac{-\mathrm{e}^{-\mathrm{i}\pi/4}}{\sqrt{2\pi k\rho\sin\beta}}\mathrm{e}^{-\mathrm{i}k(\rho\sin\beta + z\cos\beta)}\,(f_1(\pi+\varphi) - f_1(-\pi+\varphi)) + \mathrm{O}(1/(k\rho)^{3/2})$$

as $\pi - |\pm 2\Phi - (\varphi' + \varphi)| \neq 0$ and $\pi - |\varphi' - \varphi| \neq 0$, i.e. except when real poles cross the SDP. In this respect, let us note that the integral term with SDP can be evaluated asymptotically for ρ large so that the total field expression remains continuous as a pole crosses the SDP (see Kouyoumjian and Pathak (1974), Bernard (1987), Lyalinov (1994) for real poles or more generally Gennarelli and Palumbo (1984), Tuzhilin (1973a), Rojas (1988) for complex poles).

5. Conclusion

We have studied the diffraction of a skew-incident plane wave by a passive anisotropic wedge of any angle, whose principal axes of anisotropy are along and normal to the edge. A new class of canonical cases has been derived by employing the factorization technique for coupled equations, previously developed by the author.

The z-components of the field, in cylindrical coordinates (ρ, φ, z), are searched in the form of Sommerfeld–Maliuzhinets integrals as follows

$$\left| \begin{array}{l} E_z(\rho,\varphi,z) \\ H_z(\rho,\varphi,z) \end{array} \right. = \frac{\mathrm{e}^{-\mathrm{i}kz\cos\beta}}{2\pi\mathrm{i}} \int_\gamma \left| \begin{array}{c} f_1(\alpha+\varphi) \\ f_2(\alpha+\varphi)/Z_\mathrm{o} \end{array} \right. \mathrm{e}^{\mathrm{i}k\rho\sin\beta\cos\alpha}\,\mathrm{d}\alpha$$

which satisfy the wave equation $(\Delta + k^2)u = 0$. Conditions at the edge and at infinity are assumed, and we consider an anisotropic constant impedance boundary condition $(E - n^\pm(n^\pm E)) = (Z^\pm)(n^\pm \wedge H)$ on each face $\varphi = \pm\Phi$, where $n^\pm$ is the unit vector along the outward-pointing normal to the face, and $(Z^\pm)/Z_\mathrm{o}$ a relative constant impedance tensor. This implies coupled functional equations

$$A^\pm_\alpha \left| \begin{array}{l} f_1(\alpha\pm\Phi) \\ f_2(\alpha\pm\Phi) \end{array} \right. - A^\pm_{-\alpha} \left| \begin{array}{l} f_1(-\alpha\pm\Phi) \\ f_2(-\alpha\pm\Phi) \end{array} \right. = \sin\alpha \left| \begin{array}{l} c_1 \\ c_2 \end{array} \right.$$

with

$$A^\pm_\alpha = \begin{bmatrix} \cos\beta\cos\alpha & \sin\alpha \pm \eta^\pm_h \sin\beta \\ \sin\alpha \pm \sin\beta/\eta^\pm_e & -\cos\beta\cos\alpha \end{bmatrix}$$

when $(Z^\pm)/Z_\mathrm{o}$ is characterized by a class of diagonal relative impedance matrices with elements $\eta^\pm_h, \eta^\pm_e$.

By employing an original factorization technique for coupled equations, the exact closed-form solution is found when the products of these elements are equal to unity, i.e. $\eta^\pm_h \eta^\pm_e = 1$. This case of anisotropy is important. It corresponds to geometrical optics reflection coefficients which are independent of the polarization of the incident field, when the plane of incidence is perpendicular to the edge (normal incidence).

We then obtain

$$\begin{bmatrix} f_1(\alpha) \\ f_2(\alpha) \end{bmatrix} = \frac{\Psi_{an}(\alpha)}{\Psi_{an}(\varphi')} \begin{bmatrix} \cos\Delta(\alpha) & \sin\Delta(\alpha) \\ -\sin\Delta(\alpha) & \cos\Delta(\alpha) \end{bmatrix} \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} \sigma(\alpha)$$

where $\Delta(\alpha) = v_{an}(\alpha) - v_{an}(\varphi')$, with explicit closed-form expressions of the two special functions $\Psi_{an}(\alpha)$ and $v_{an}(\alpha)$.

A decomposition of the field in terms of geometrical optics field and of waves specifically excited by the edge are specified.

Appendix 1

Let us consider a harmonic electromagnetic field with a z dependence $\mathrm{e}^{-\gamma z}$, $\gamma = \mathrm{i}k\cos\beta$ (time convention $\mathrm{e}^{\mathrm{i}\omega t}$). Let $\epsilon_\mathrm{o}, \mu_\mathrm{o}, k$, be the permittivity, the permeability and the associated wavenumber of the exterior medium of propagation. The application of the Maxwell equations leads us to the expressions of the Cartesian components of the electric and magnetic fields E and H (see in particular Jones (1964)) according to

$$E_x = \frac{-\gamma}{\gamma^2+k^2}\frac{\partial E_z}{\partial x} - \frac{\mathrm{i}\omega\mu_o}{\gamma^2+k^2}\frac{\partial H_z}{\partial y} \tag{A1.1}$$

$$E_y = \frac{-\gamma}{\gamma^2+k^2}\frac{\partial E_z}{\partial y} + \frac{\mathrm{i}\omega\mu_o}{\gamma^2+k^2}\frac{\partial H_z}{\partial x} \tag{A1.2}$$

$$H_x = \frac{-\gamma}{\gamma^2+k^2}\frac{\partial H_z}{\partial x} + \frac{\mathrm{i}\omega\epsilon_o}{\gamma^2+k^2}\frac{\partial E_z}{\partial y} \tag{A1.3}$$

$$H_y = \frac{-\gamma}{\gamma^2+k^2}\frac{\partial H_z}{\partial y} - \frac{\mathrm{i}\omega\epsilon_o}{\gamma^2+k^2}\frac{\partial E_z}{\partial x}. \tag{A1.4}$$

Appendix 2

Let us show that each z-component E_z and H_z tends to a finite value independent of φ as $\rho \to 0$, and the other components are locally summable in ρ in the vicinity of the origin, when we have: (b) some constants $g_j^{\pm}$ and some analytic function h_j exist such as $|f_j(\alpha+\varphi) \mp f_j(-\alpha+\varphi) - g_j^{\pm}| < |h_j(\alpha)|$ on and within some γ_+, for $|\mathrm{Re}\,\varphi| \leqslant \Phi$, and as h_j is summable on γ_+ and regular on and within it.

(a) Each z-component E_z and H_z tends to a finite value when $\rho \to 0$:

For the sake of simplicity, let us name u, the components E_z or $Z_o H_z$, and omit the index j for the function of α. We have, from (2),

$$u = \frac{\mathrm{e}^{-\mathrm{i}kz\cos\beta}}{2\pi\mathrm{i}}\int_{\gamma_+}(f(\alpha+\varphi) - f(-\alpha+\varphi))\mathrm{e}^{\mathrm{i}k\rho\sin\beta\cos\alpha}\,\mathrm{d}\alpha \tag{A2.1}$$

with γ_+, a loop above all the singularities of the integrand from $(\mathrm{i}\infty + \arg(\mathrm{i}k) + (a_1+\pi/2))$ to $(\mathrm{i}\infty + \arg(\mathrm{i}k) - (a_2+\pi/2))$ with $\mathrm{Im}\,\alpha > d$. Let us write

$$f(\alpha+\varphi) - f(-\alpha+\varphi) = [g^+] + [f(\alpha+\varphi) - f(-\alpha+\varphi) - g^+] \tag{A2.2}$$

in the previous expression. We can then consider *each integral* related to *each term within brackets* and prove they tend to a constant when $\rho \to 0$.

For the *first* integral, we can proceed by two different methods:

- we can deform the path γ and take $a_1 + a_2 = \pi$, so that, by periodicity, we obtain that the only contribution comes for a finite segment at $\mathrm{Im}\,\alpha = d$ of length 2π;
- or equivalently, we can use the expression of the Bessel function

$$J_\nu(z) = \frac{\mathrm{e}^{-\mathrm{i}\nu\pi/2}}{2\pi}\int_{\gamma_+}\mathrm{e}^{\mathrm{i}z\cos\alpha}\mathrm{e}^{\mathrm{i}\nu\alpha}\,\mathrm{d}\alpha \tag{A2.3}$$

so that we obtain

$$\lim_{\rho\to 0}\frac{1}{2\pi\mathrm{i}}\int_{\gamma_+} g^+\mathrm{e}^{\mathrm{i}k\rho\sin\beta\cos\alpha}\,\mathrm{d}\alpha = \mathrm{i}g^+J_o(0) = \mathrm{i}g^+. \tag{A2.4}$$

As regards to the *second* integral, from our assumptions in (b), it is absolutely convergent for $\rho = 0$ and then tends to a constant as $\rho \to 0$. Moreover, the functions h_j being regular within γ_+ and tending to 0 at infinity means that the integral attached to it vanishes as $\rho \to 0$, since we can deform γ_+ taking $d \to \infty$ as $\rho \to 0$ while keeping the term $\exp(\mathrm{i}k\rho\sin\beta\cos\alpha)$ bounded on the path.

Finally, the proof that u tends to a finite value ($= \mathrm{i}g^+$ at $z = 0$) independent of φ as $\rho \to 0$ has been given.

(b) Other components are summable in ρ in the vicinity of the origin:

From the appendix 1, any other component of the fields, that we now name indifferently v, is a linear combination of

$$\frac{\partial u}{\partial \rho} = \frac{\mathrm{e}^{-\mathrm{i}kz\cos\beta}}{2\pi\mathrm{i}} \int_{\gamma_+} \mathrm{i}k\sin\beta\cos\alpha(f(\alpha+\varphi) - f(-\alpha+\varphi))\mathrm{e}^{\mathrm{i}k\rho\sin\beta\cos\alpha}\,\mathrm{d}\alpha \tag{A2.5}$$

and

$$\frac{\partial u}{\rho\partial \varphi} = \frac{\mathrm{e}^{-\mathrm{i}kz\cos\beta}}{2\pi\mathrm{i}} \int_{\gamma_+} \mathrm{i}k\sin\beta\sin\alpha(f(\alpha+\varphi) + f(-\alpha+\varphi))\mathrm{e}^{\mathrm{i}k\rho\sin\beta\cos\alpha}\,\mathrm{d}\alpha. \tag{A2.6}$$

Then, we let

$$f_j(\alpha+\varphi) \mp f_j(-\alpha+\varphi) = [g^{\pm}] + [f_j(\alpha+\varphi) \mp f_j(-\alpha+\varphi) - g_j^{\pm}] \tag{A2.7}$$

since we have

$$\frac{-\mathrm{e}^{-\mathrm{i}n\pi/2}}{2\pi} \int_{\gamma_+} \mathrm{e}^{\mathrm{i}n\alpha}\mathrm{e}^{\mathrm{i}k\rho\cos\alpha}\,\mathrm{d}\alpha = J_n(k\rho) \tag{A2.8}$$

which is $\mathrm{O}(k\rho)$ as $\rho \to 0$ when $|\mathrm{Re}\,(n)| = 1$, it is clear that the integral attached to the term $[g^{\pm}]$ of (A2.7), for the expression of v, is $\mathrm{O}(k\rho)$.

Then, using the bound we have, from condition (b), on the second term within brackets in (A2.7), we can write

$$|v(\rho)| < B \int_{\gamma_+} |\cos\alpha||h(\alpha)||\mathrm{e}^{\mathrm{i}k\rho\sin\beta\cos\alpha}||\,\mathrm{d}\alpha| + \mathrm{O}(k\rho) \tag{A2.9}$$

B being some constant independent of ρ, and h being summable on γ_+.

Now, since we have

$$\mathrm{e}^{\mathrm{i}k\rho\sin\beta\cos\alpha} = \mathrm{O}(\mathrm{e}^{-|\mathrm{i}k|\rho\sin\beta\sin(a_{\min})\cosh(\mathrm{Im}\,\alpha)})$$

at infinity on γ_+, with $a_{\min} = \min(a_1, a_2) > 0$, and since h is summable on γ_+, the integral term present in (A2.9) is locally summable with respect to ρ in the vicinity of $\rho = 0$, and so, finally, $v(\rho)$ is locally summable with respect to ρ in this region, which achieves the proof.

Appendix 3. The inversion theorem of Maliuzhinets and the Sommerfeld–Maliuzhinets transform

We give here details concerning the inversion theorem of Maliuzhinets and the Maliuzhinets transform.

Two theorems published by Maliuzhinets (1958) establish the basis of this transform. Too often, in applications, the second theorem is neglected or the first theorem is given in a partial form; we prefer to give them in their original form.

Theorem 1. Let M, a, b, c, d be positive numbers; let ϵ, m be numbers satisfying the conditions: $0 < \epsilon < \pi$, $|\arg(m)| \leqslant \pi/2$. Given the integral equation

$$F(r) = \frac{1}{2\pi\mathrm{i}} \int_{\gamma} \mathrm{e}^{mr\cos\alpha} f(\alpha)\,\mathrm{d}\alpha. \tag{A3.1}$$

The given function $F(r)$ satisfies the inequality $|F(r)| < M|r|^{-1+a}\mathrm{e}^{b|r|}$ for positive values of r and also in the entire region $c < |r| < \infty$, $|\arg(r)| < \epsilon_1$, where this function is analytic and regular.

The contour of integration γ is made of two loops. The loop γ_1 consists of the half lines $\operatorname{Re}\alpha = \arg(m) \pm (\epsilon + \pi/2)$, $\operatorname{Im}\alpha \geqslant d$ and the line segment $\operatorname{Im}\alpha = d$. The loop γ_2 is symmetric to γ_1 with respect to inversion in the origin $\alpha = 0$.

Then, among those analytic functions $f(\alpha)$, which are regular on the contour γ and within both loops except possibly at infinitely distant points, and which satisfy in these regions the inequality $|f(\alpha)| < M_1 \exp[(1 - a_1)|\operatorname{Im}\alpha|]$, there exists one and only one odd function which is a solution of the integral equation (A3.1). For $\operatorname{Re}(m\cos\alpha) > b$ this function is represented by the integral

$$f(\alpha) = -\frac{m\sin\alpha}{2}\int_{\to 0}^{\to\infty} F(r)\mathrm{e}^{-mr\cos\alpha}\,\mathrm{d}r \tag{A3.2}$$

for this function $a_1 = a$.

Theorem 2. Let $f(\alpha)$ be an analytic function, regular on the contour γ and in the interior of the loops γ_1 and γ_2 of theorem 1 everywhere, except for infinitely distant points. For $|\operatorname{Im}\alpha| \to \infty$, let $f(\alpha) = \mathrm{O}(\exp[(n + 1 - a)|\operatorname{Im}\alpha|])$ in these regions, where $0 < a < 1$ and n is a positive integer or zero.

The, in order for

$$\frac{1}{2\pi\mathrm{i}}\int_{\gamma} \mathrm{e}^{mr\cos\alpha} f(\alpha)\,\mathrm{d}\alpha = 0 \tag{A3.3}$$

to hold for $r > 0$, it is necessary and sufficient that the functions $f(\alpha)$ have the form

$$f(\alpha) = f_1(\alpha) + \sin\alpha\sum_{\nu=0}^{n} \nu c_\nu \cos^{\nu-1}\alpha \tag{A3.4}$$

where $f_1(\alpha)$ is an arbitrary even function and the coefficients c_ν are arbitrary constants; or, as follows from (A3.4), that the functions $f(\alpha)$ satisfy the functional equation

$$f(\alpha) - f(-\alpha) = 2\sin\alpha\sum_{\nu=0}^{n} \nu c_\nu \cos^{\nu-1}\alpha.$$

For the proof of the last theorem, Maliuzhinets remarks that it is possible to consider that $F(r)$ is $\mathrm{O}(r^{-n+a})$ instead of $\mathrm{O}(r^{-1+a})$ as $r \to 0$, by taking in theorem 1: $r^{n-1}F(r) \leftrightarrow F(r)$, $D^{n-1}(f) \leftrightarrow f$ with $D(f) = \frac{1}{m}\frac{\partial}{\partial\alpha}\left(\frac{f(\alpha)}{\sin\alpha}\right)$, and then integrating $D^{n-1}(f) = 0$.

Notes.

• For any function regular as $|\operatorname{Im}\alpha| > d$ and having a period 2π, the semi-infinite lines of γ with $|\operatorname{Im}\alpha| > d$, when defined with $\epsilon = \pi/2$, give no contribution to the integral A3.1, by mutual cancellation.

• We have, for $|\arg(z)| < \pi/2$ (see Gradshtein (1980)),

$$J_\nu(z) = \frac{-\mathrm{e}^{-\mathrm{i}\nu\pi/2}}{2\pi}\int_{\gamma_1} \mathrm{e}^{\mathrm{i}z\cos\alpha}\mathrm{e}^{\mathrm{i}\nu\alpha}\,\mathrm{d}\alpha \tag{A3.5}$$

and then, from $H_\nu^{(2)}(z) = \mathrm{i}(J_{-\nu}(z) - \mathrm{e}^{\mathrm{i}\nu\pi}J_\nu(z))/\sin(\nu\pi)$

$$H_\nu^{(2)}(z) = \frac{-\mathrm{e}^{\mathrm{i}\nu\pi/2}}{2\pi\mathrm{i}\sin(\nu\pi)}\int_{\gamma} \mathrm{e}^{\mathrm{i}z\cos\alpha}\mathrm{e}^{\mathrm{i}\nu\alpha}\,\mathrm{d}\alpha.$$

• In the previous theorems, we assumed that $F(\rho)$ is analytical and regular in the region $c < |\rho| < \infty, |\arg(\rho)| < \epsilon_1$, in order to have the integral expression of f regular within γ for $\epsilon = \epsilon_1 > 0$. If we can replace the dependence with respect to ρ and k by the one with respect to $k\rho$, we can then obtain the latter property from the regularity properties of

F in some vicinity of k. Let us note that when $F(\rho)$ is a field, the regularity is given for $\mathrm{Im}\,\omega < 0(\mathrm{Re}\,\mathrm{i}k > 0, k/\omega$ being real when there is no loss in the medium of propagation) from the causality principle.

Appendix 4

Let us show here how (11*c*–*d*) are equivalent to the equalities furnished by the diagonal members of (10*a*), when (11*a*) and (11*b*) are satisfied.

We can write the equalities on the diagonal elements in (10*a*) as follows

$$\frac{1}{\det A_\alpha} g\left(\frac{a_+}{c_+}, \frac{d_-}{b_-}\right) = -\frac{\varepsilon}{c_+ b_-}(a_+ b_+ - c_+ d_+) \tag{A4.1}$$

$$\frac{1}{\det A_\alpha} g\left(\frac{d_+}{b_+}, \frac{a_-}{c_-}\right) = \frac{\varepsilon}{c_- b_+}(a_+ b_+ - c_+ d_+). \tag{A4.2}$$

If one divides (A4.1) by (A4.2), one obtains (11*c*):

$$\left(\frac{b_+}{c_+}\right)\left(\frac{b_-}{c_-}\right)^{-1} = \frac{-g(a_+/c_+, d_-/b_-)}{g(d_+/b_+, a_-/c_-)}.$$

The term $g(d_+/b_+, a_-/c_-)$ changes into $-g(a_+/c_+, d_-/b_-)$ when α changes into $-\alpha$, so that the equation (A4.2), modified by taking the argument $-\alpha$, then divided by (A4.1), gives us (11*d*):

$$\frac{(b_+ a_+)}{(b_- a_-)} = \frac{\det A_{-\alpha}}{\det A_\alpha} \frac{1 - (c_-/a_-)(d_-/b_-)}{1 - (c_+/a_+)(d_+/b_+)}.$$

Inversely, let us show now that, when (11*a*) and (11*b*) are satisfied, (11*c*) and (11*d*) are sufficient to recover (A4.1) and (A4.2).

As a matter of fact, writing

$$\frac{1}{\det A_\alpha} g\left(\frac{a_+}{c_+}, \frac{d_-}{b_-}\right) = -\frac{w_1(\alpha)}{c_+ b_-}(a_+ b_+ - c_+ d_+) \tag{A4.3}$$

$$\frac{1}{\det A_\alpha} g\left(\frac{d_+}{b_+}, \frac{a_-}{c_-}\right) = \frac{w_2(\alpha)}{c_- b_+}(a_+ b_+ - c_+ d_+) \tag{A4.4}$$

we have (11*c*) that implies

$$w_1 = w_2 = w$$

and then (11*d*) which gives us

$$w(\alpha) = w(-\alpha)$$

which leads us to (A4.1) and (A4.2) with ε replaced by the function w.

Therefore, when (11*a*–*d*) are satisfied, an equation, similar to (8), where ε is replaced by w, is satisfied, and so we have

$$(C_{\alpha\pm\Phi})(A_\alpha^\pm)^{-1}(A_{-\alpha}^\pm)(C_{-\alpha\pm\Phi})^{-1} = w^\pm(\alpha)\mathrm{Id} \tag{A4.5}$$

where Id is the identity matrix, w being denoted $w^\pm$ for each face.

Now, if one changes the sign of α in the last equation (A4.5), one obtains the inverse of the left member and thus

$$w(-\alpha) = 1/w(\alpha).$$

Since $w(\alpha)$ is even, this leads us to

$$w(\alpha) = +1 \text{ or } -1 \tag{A4.6}$$

which completes the demonstration.

Note that if $(C_{\alpha\pm\Phi})(A_\alpha^\pm)^{-1}|_{\alpha=0}$ and their inverse are definite, then $w(\alpha) = 1$.

Appendix 5

Let us explain why, when n is not (the function) zero (i.e. in the coupled case), c and b are not zero if $\det((C_\alpha)^{-1}) \neq 0$.

Assuming that the function n is not (the function) zero, we deduce, from the nondiagonal terms of (10a), that the function c (resp. b) cannot be zero, except if a (resp. d), and then $\det((C_\alpha)^{-1})$, would be zero. Note that a comparable reasoning is possible when we assume p is not zero.

Appendix 6

We give here the solution of $s(\alpha \pm \Phi) - \varepsilon s(-\alpha \pm \Phi) = S^\pm(\alpha)(\varepsilon = + \text{ or } -1)$, when the function $S^\pm$ is regular on the imaginary axis, exponentially decreasing as $|\mathrm{Im}\,\alpha| \to \infty$, and s is regular as $|\mathrm{Re}\,\alpha| \leqslant \Phi$.

Let us consider, the functional equations for s

$$s(\alpha \pm \Phi) - \varepsilon s(-\alpha \pm \Phi) = S^\pm(\alpha) \tag{A6.1}$$

ε being $+1$ or -1, the functions $S^\pm$ being analytic on the imaginary axis and $\mathrm{O}(\exp(-a|\mathrm{Im}\,\alpha|))$, $a > 0$, as $|\mathrm{Im}\,\alpha| \to \infty$. Let us search the solution s, regular as $|\mathrm{Re}\,\alpha| \leqslant \Phi$ (even at infinity), as follows

$$s(\alpha) = \frac{s(\mathrm{i}\infty) + s(-\mathrm{i}\infty)}{2} + \frac{s(\mathrm{i}\infty) - s(-\mathrm{i}\infty)}{2}\,\mathrm{sign}(\mathrm{Im}\,\alpha) + s_{\mathrm{o}}(\alpha) \tag{A6.2}$$

where s_{o} is absolutely integrable on any line $\mathrm{Re}\,\alpha = \alpha_{\mathrm{o}}$ as $|\alpha_{\mathrm{o}}| \leqslant \Phi$. We will have then $s(\mathrm{i}\infty) - \varepsilon s(-\mathrm{i}\infty) = 0$. This type of equation has been solved, in particular in the works of Maliuzhinets, Tuzhilin, Bernard, for the analytical determination of the solutions of wedge diffraction problems.

Let us take the Fourier transform of (A6.1). We obtain a system of two equations:

$$\mathcal{F}(\omega)\mathrm{e}^{\mp\mathrm{i}\omega\Phi} - \varepsilon\mathcal{F}(-\omega)\mathrm{e}^{\pm\mathrm{i}\omega\Phi} = \mathcal{H}^\pm(\omega). \tag{A6.3}$$

Multiplying them by $\exp(\mp\mathrm{i}\omega\Phi)$ and finding the difference, we obtain that the function $\mathcal{F}$, the Fourier transform of s, follows

$$2\mathrm{i}\mathcal{F}(\omega)\sin(2\omega\Phi) = \mathcal{H}^-(\omega)\mathrm{e}^{\mathrm{i}\omega\Phi} - \mathcal{H}^+(\omega)\mathrm{e}^{-\mathrm{i}\omega\Phi}.$$

The analytic function obtained by dividing the right member of the previous equality with $2\mathrm{i}\sin(2\omega\Phi)$ does verify (A6.3), and then, the inverse Fourier transform (integral taken in the sense of principal value) of this expression satisfies (A6.1) for $|\mathrm{Re}\,\alpha| < \Phi$. So, we have the developed form of s, for $|\mathrm{Re}\,\alpha| < \Phi$,

$$s(\alpha) = \frac{s(\mathrm{i}\infty) + s(-\mathrm{i}\infty)}{2} + \frac{\mathrm{i}}{\sqrt{2\pi}}\ \mathrm{v.p}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\left(\frac{R_+(\omega)\mathrm{e}^{-\mathrm{i}\omega\Phi}}{\mathrm{i}\sin(2\omega\Phi)} - \frac{R_-(\omega)\mathrm{e}^{\mathrm{i}\omega\Phi}}{\mathrm{i}\sin(2\omega\Phi)}\right)\mathrm{e}^{-\mathrm{i}\omega\alpha}\,\mathrm{d}\omega \tag{A6.4a}$$

with

$$R_{\pm}(\omega) = \frac{-\mathrm{i}}{2\sqrt{2\pi}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} S^{\pm}(\alpha')\mathrm{e}^{\mathrm{i}\omega\alpha'}\,\mathrm{d}\alpha' \tag{A6.4b}$$

where the quantity $s(\mathrm{i}\infty)+s(-\mathrm{i}\infty)$ is arbitrary when $\varepsilon = 1$ and equal to zero when $\varepsilon = -1$. The term v.p. means that we take the principal value of the integral.

For the case where the function s is v'_{an} or $(\ln(\Psi_{an}))'$, $R_{\pm}$ can be evaluated explicitly by the method of residues. Besides, we have $\varepsilon = -1$ so $s(\mathrm{i}\infty) + s(-\mathrm{i}\infty) = 0$. So, we obtain a simple integral expression of v_{an} and Ψ_{an}, corresponding to (16a) and (21a).

Since the functions $S^{\pm}$ are absolutely integrable, it is possible to simplify the form (A6.4a) by interchanging the order of integration, so that we obtain as in Tuzhilin (1973b) and Bernard (1990a), for $|\mathrm{Re}\,\alpha| < \Phi$,

$$\begin{aligned} s(\alpha) = \frac{s(\mathrm{i}\infty)+s(-\mathrm{i}\infty)}{2} + \frac{-\mathrm{i}}{8\Phi}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \mathrm{d}\alpha' \Bigg(S^{+}(\alpha')\tan\left(\frac{\pi}{4\Phi}(\alpha+\Phi-\alpha')\right) \\ -S^{-}(\alpha')\tan\left(\frac{\pi}{4\Phi}(\alpha-\Phi-\alpha')\right)\Bigg). \end{aligned} \tag{A6.5}$$

Considering $S^{\pm} = \mathrm{O}(\exp(-a|\mathrm{Im}\,\alpha|))$ and $\varepsilon = -1$, and developing the terms tan(.) in (A6.5), we obtain that

$$s(\alpha) = \frac{\pm 1}{8\Phi}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \mathrm{d}\alpha'(S^{+}(\alpha') - S^{-}(\alpha')) + \mathrm{O}(\mathrm{e}^{-c|\mathrm{Im}\,\alpha|})$$

as $\mathrm{Im}\,\alpha \to \pm\infty$, with $c > 0$ being some constants independent of α.

For $(v_{an})'$ and $(\ln\Psi_{an})'$, the terms $S^{\pm}$ are the derivatives of some known analytic functions so that the integral term can be determined directly. Then, for $s = (v_{an})'$, we obtain $v_{an}(\alpha) = A_1 + \mathrm{O}(\mathrm{e}^{-c|\mathrm{Im}\,\alpha|})$, and, for $s = \ln(\Psi_{an})'$, $\Psi_{an}(\alpha) = A_2\exp(\mu|\mathrm{Im}\,\alpha|)(1 + \mathrm{O}(\mathrm{e}^{-c|\mathrm{Im}\,\alpha|}))$ with $\mu = \pi/2\Phi$, A_1 and A_2 being some constants. This behaviour, deduced from an expression valid for $|\mathrm{Re}\,\alpha| < \Phi$, remains valid in any band $|\mathrm{Re}\,\alpha| <$ constant, by the argument of continuation (from (A6.1) or from the deformation of the integration path in (A6.5)).

Notes.

- As Maliuzhinets found,

$$\mathrm{v.p.}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \equiv \frac{1}{2}\left(\int_{-\mathrm{i}\infty-\epsilon_1}^{\mathrm{i}\infty-\epsilon_1} + \int_{-\mathrm{i}\infty+\epsilon_1}^{\mathrm{i}\infty+\epsilon_1}\right)$$

with $\epsilon_1 < a$.

- The expression (A6.5) can be continued analytically for $|\mathrm{Re}\,\alpha| \geqslant \Phi$ by considering the residue due to the poles crossing the path of integration, so that we verify easily that the continuation of this expression satisfies the initial functional equations. This latter expression continues to satisfy the functional equation when the analytic function $S^{\pm}$, regular on the imaginary axis, is only assumed to be absolutely integrable.

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